# Analysis and Optimal Design of Discrete Order Picking Technologies Along a Line

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**Abstract:** Order picking accounts for most of the operating expense of a typical distribution center, and thus is often considered the most critical function of a supply chain. In *discrete* order picking a single worker walks to pick all the items necessary to fulfill a single customer order. Discrete order picking is common not only because of its simplicity and reliability, but also because of its ability to pick orders quickly upon receipt, and thus is commonly used by e-commerce operations. There are two primary ways to reduce the cost (walking distance required) of the order picking system. First is through the use of *technology*—conveyor systems and/or the ability to transmit order information to pickers via mobile units. Second is through the *design*—where best to locate depots (where workers receive pick lists and deposit completed orders) and how best to lay out the product. We build a stochastic model to compare three configurations of different technology requirements: single-depot, dual-depot, and no-depot. For each configuration we explore the optimal design. © 2008 Wiley Periodicals, Inc. Naval Research Logistics 00: 000–000, 2008

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## 1. INTRODUCTION

Within the warehouse, the order picking function typically accounts for about 55% of operating costs (see [7]). In general, order picking is commonly considered the most critical function in a supply chain (see for instance [17]). The importance of order picking is becoming more apparent as new e-commerce operations struggle to compete with traditional bricks-and-mortar operations. Consider for example e-commerce grocery services (such as Webvan or Peapod). They distinguish themselves from traditional grocery stores in that they must absorb the cost to pick customer orders, whereas for a traditional grocery store the customer performs this function for free.

In *discrete* order picking a single worker picks all items necessary to fulfill a single customer order, and picks no other items until the order for the customer is complete. This method of order picking is common because it is simple and reliable in that a picker need to manage only one customer order at a time. Furthermore, a customer order is picked quickly upon receipt without delaying to batch with other customer orders or to hand off a partially picked order from one picker to another; and therefore, discrete order picking is commonly used for real-time operations. For instance,

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discrete order picking is used at some discount stores such as Service Merchandise, where essentially all items for a customer order are picked from the in-store warehouse while the customer waits.

Figure 1 depicts an implementation of discrete order picking along a linear pick line. The picker retrieves a printed pick list for a single order at the *depot*, located at the beginning of the line, along with an empty bin to carry the items. He walks as far down the line as necessary to retrieve all the items ordered by the customer, and then returns to the depot to deposit the picked items and retrieve a new pick list and empty bin. Other workers may also be simultaneously and independently picking orders.

One implementation of discrete order picking along a line, in which the author has been involved, is at Urbanfetch.com, an e-commerce retailer that delivered a large variety of items to customers within an hour. Discrete order picking is the natural picking discipline for Urbanfetch. Once an order is received it must be picked quickly. A worker retrieves the pick list at the depot, walks to retrieve all items, and then returns the items to the depot where they await dispatch for delivery. Management is able to easily adjust picking capacity on the fly, moving workers to and from restocking or other areas when needed (even senior management picks orders when necessary). Thus the advantages for Urbanfetch is that the picking protocol is simple, reliable, scalable, and responsive.
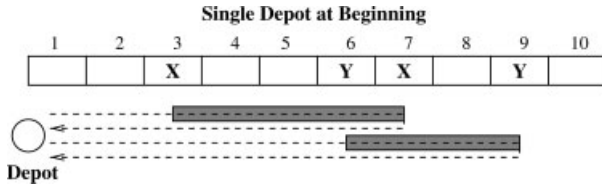
**Single Depot at Beginning**



**Figure 1.** Two customer orders, X and Y. Order X has its leftmost pick at location 3, and its rightmost pick at 7. Order Y has its leftmost pick at 6, and its rightmost pick at 9. With a single-depot located at the beginning of the line the picker walks along the line until the order is picked. The shaded portion depicts the part of the walk path from when the first item for the order is picked until the last.

Since it is advantageous for workers to avoid the waste associated with walking between aisles or across conveyor systems, discrete order picking along a line is common. Furthermore, discrete order picking along a line acts as a fundamental building block of other more elaborate picking schemes. Consider for example a *zone* picking system as shown in Fig. 2. Here, five workers are each assigned to their own zones (for simplicity each zone is shown as its own entire pick aisle, but it is common for a worker zone to cover only a portion of an aisle). Each picker remains in his zone, picking just the items that are necessary for each customer order. When his picks are complete within his zone he deposits the items on the conveyor which might then be carried to a consolidation area before eventually arriving at the shipping department. (See [17] for an overview of order picking protocols including discrete and zone picking; and [1] for an implementation where each worker picks a portion of a pick line.)

The main disadvantage of discrete order picking is that the amount of walking per pick can be high. Multiple customer orders can, however, be combined into a batch (*batch* picking) to help reduce the walking per pick; and our models
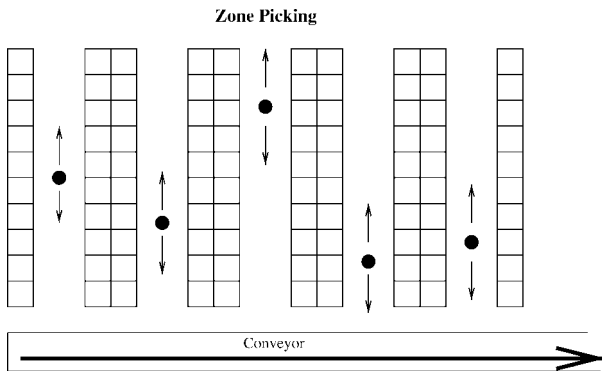
equally apply to such batch picking operations. Our focus here is to examine how to configure the system to minimize the amount of walking. We do not consider the time to actually pick an item once the picker is in position, since this time is required (fixed) regardless of the configuration. But we note that technologies such as pick-to-light systems can speed up this fixed portion of the efforts of the picker equally well for all of our configurations.

We consider two primary ways in which one can reduce the walking required for a discrete order picking operation.

**Technology:** By the technology we mean the way in which the picker receives picking information (the pick list) and the way in which he deposits the items after they are picked. He may receive the pick list in print form at a depot, or more technology can be used so that the pick information is transmitted to him via a mobile device, such as with RF (radio frequency) technology. The picked items of an order may be deposited by walking them back to a depot location; or with conveyor technology the completed order can be deposited anywhere along the pick line onto a conveyor system.

**Design:** We consider two design issues. First, where to locate the depot(s)—if any are used. And second, how best to assign items to locations.

We examine three **configurations** that each utilize a different level of technology:

- **Single-depot**: with no technology investment,
- **Dual-depots**: with conveyor technology investment, and
- **No-depot**: with conveyor and RF technology investment.

The first configuration is the simplest and most common— a *single depot* at location $k$. Figure 1 shows a common case where the depot is located at the start of the line, $k = 1$. Figure 3 shows a case when a single depot is located within

**Zone Picking**



**Figure 2.** Each of five workers remain in their zones (in this case an aisle). Here, this zone picking operation is composed of five, independent, discrete picking operations.
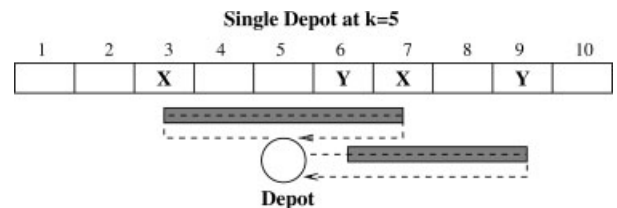
**Single Depot at k=5**



**Figure 3.** Two customer orders, X and Y. With a single depot located at $k = 5$ the picker first walks to the leftmost pick of order X, then along the line picking order X until the rightmost pick for order X is complete. Then he returns to the depot and repeats the procedure for order Y.

the pick line. The picker begins each order by retrieving a pick list from the depot and walking to the leftmost pick, and then picking all the items in sequence until the rightmost item is picked, and then walking back to deposit the items at the depot. It is simple to convince oneself that it is never better to alter such a sequence of picks by picking an item central to the order first.

Our next configuration introduces a new design to discrete order picking—*dual depots* located at $u$ and $v$ (see Fig. 4). We have never seen this in practice, except when the two depots are located at opposite ends of the pick line. The protocol is that a worker would first pick up a pick list at the left depot, walk to the left-most pick, and then pick the order with a left-to-right pass along the line. He would then visit the right depot to deposit the current order and retrieve a pick list for a new order, and then make a right-to-left pass along the aisle to pick the next order. Finally, he returns to the left depot to deposit the current order and retrieve a new pick list. A centralized computer might simply alternate printing pick lists between the two depots, and when printing a pick list to the leftmost (rightmost) depot it prints the items in a left-right (right-left) sequence so picking is easy. For many implementations this configuration might require a conveyor to transport picked orders from the two depots to shipping.

Finally we consider the best possible discrete picking implementation, a *no-depot* configuration, that requires full technology—eliminating the need to walk to and from depot locations. A conveyor is installed so that a picker can deposit a completed order anywhere along the pick aisle, and an overhead rail or rack under the conveyor is used to store empty bins for new orders. The picker has a mobile unit that transmits pick information. This is commonly accomplished using an RF device, which is often strapped like a large watch on the wrist. Thus, whenever a worker completes an order he immediately deposits the completed order on the conveyor, collects an empty bin, and then pushes a button on his RF
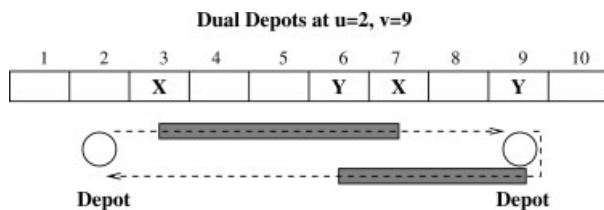


**Figure 5.** Two customer orders, X and Y. With no depot the picker starts at the leftmost pick of order X, then walks along the line picking order X until the rightmost pick is complete, and then deposits order X along the conveyor. He then walks to the rightmost pick of order Y, then walks along the line picking order Y until the leftmost pick is complete, and then deposits order Y on the conveyor.

device to display the picks necessary for the next order (see Fig. 5).

We build a stochastic model of the discrete order picking operation. For each configuration we determine the expected amount of walk distance required per order. We do not model the time required to stop and pick each item, since this is the same regardless of the configuration—the configurations differ only in the distance walked per order. Our model is useful in ascertaining the benefit of technology—an extra depot, a conveyor, or RF devices. We will find that the benefit depends on whether the order stream has many pick locations required per order or very few.

Next we consider design issues for the single- and dual-depot configurations. First, we will consider where best to locate single or dual depots given a product layout. This is a practical design issue when a depot is simply a printer to output pick lists and a rack to hold picked orders and thus can easily be moved. However, sometimes the depot locations correspond to shipping docks or the front door of an e-commerce distribution center. In such cases we investigate how best to layout the product when the depot location(s) are fixed. We also examine how to layout items for the no-depot configuration. Finally, we solve the full design question for single and dual depots—how best to both locate depots and layout product simultaneously.



**Figure 4.** Two customer orders, X and Y. With dual-depots located at $u = 2$ and $v = 9$, the picker starts at the left depot and walks to the leftmost pick of order X, then walks along the line picking order X until rightmost pick is completed. He then returns to the right depot, and then walks to the rightmost pick of order Y, then walks along the line picking order Y until the leftmost pick is complete for order Y, before returning to the left depot for the next order.
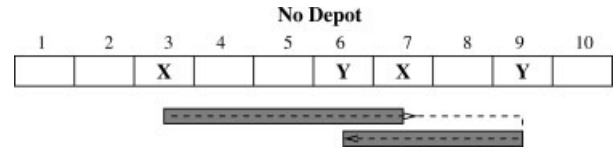
## 2. OUR MODEL

We model a stochastic stream of orders in which the makeup of each order is independently and identically distributed. We divide the pick line into $n$ locations or regions. Each order is represented as an $n$-tuple, the $i$-th entry of which is $p_i$ — the probability of at least one pick occurring at location $i$, and is independent of other locations. For instance, if the pick line is composed of 20 static shelves, we might model an order by setting $n = 20$. Then for each order, the probability of at least one pick required from shelf

$i$ is $p_i$, and the probability that shelf $i$ requires no pick is $q_i = 1 - p_i$.

Our model does not explicitly handle correlation among the items. However, it is common in practice for strongly correlated items to be located together. For example at Revco Drugs, products are stored for picking in the distribution center in the same way as retail stores are laid out in order to facilitate restocking at the retail stores (see [1]). Therefore, for example, while there is certainly considerable correlation between different shampoos; they are located in the same rack and so can be represented by the same $p_i$. Similar items are also often shelved together because of their physical attributes. At Urbanfetch.com different shelving is required for snack chips, video tapes, and ice cream. So, in many practical settings, we can expect our model to handle some types of correlation well.

We assume that the distance between adjacent pick locations is the same and of unit length. This simplifies our notation with little sacrifice to the robustness of our model. If different distances between locations is needed then dummy locations can be inserted with $p_i$'s set to 0. Thus, if a picker travels from location $i$ to location $j$ then he walks a distance of $j - i$ units.

## 3.  RELATED WORK

The closest work in the literature appears to be Jarvis and McDowell [9] who also consider a discrete order picking system in which items have an independent pick probability. They concentrate on the product layout design problem for a single-depot configuration in which a single picker navigates parallel aisles (a rectangular warehouse) as in Fig. 2. They assume that once a picker enters a vertical aisle he must traverse the entire aisle. Thus the heart of their model concerns how far a picker must walk along the horizontal line, which is the same as our single-depot model. They determine the best product layout for two special cases of a single-depot configuration—a depot at the beginning of the line or one in the center. Our work for the single-depot model generalizes their work by considering a single depot anywhere along the pick line. We solve the full design problem of how to simultaneously layout items and locate the depot.

Jewkes et al. [10] consider multiple pickers on a line in which an order is passed from the zone of one worker to another. They consider the simultaneous problem of product location, single depot location for each picker, and zone formation. Each picker within their zone is similar to our single depot configuration, but their model of picker movement is greatly simplified by assuming that only one pick location is visited for each pick list retrieved at the depot (see also [19]). Bartholdi et al. [1] examine bucket brigade order picking, in which a team of workers combine efforts along a pick line to pick customer orders with all zones eliminated.

Much of the other work in which pickers are routed to items considers batch picking—so that there is a fixed number of picks (the batch size) per route. Chew and Tang [5] and Le-Duc and de Koster [12] examine the effects of batch sizes in a real time system for rectangular warehouses. With the number of picks per order fixed, they develop a queueing model where incoming orders are batched before picking. Rosenwein [16] also examines the order batching problem.

Roodbergen and Vis [15] show that for batch picking and uniformly distributed picks in a rectangular warehouse, the best depot location is in the middle of the front cross-aisle. Le-Duc and de Koster [11] consider items zoned by storage classes within a rectangular warehouse. Picks within a class are uniformly distributed and the total number of picks is fixed. They develop a stochastic model to estimate zone sizes and travel distances.

Caron et al. [3] develop a stochastic model of the expected walking distance for different routing strategies in a rectangular warehouse. While the number of picks per order is fixed, the items are stored based on their order frequency. Caron et al. [4] extends this work to examine how to design the number of aisles in a warehouse.

Another stream of literature considers the walking distance required to pick an order in a rectangular warehouse; Ratliff and Rosenthal [14] solve this special case of the traveling salesman problem, and Hall [8] considers the expected route lengths in a rectangular warehouse for a variety of layouts and strategies.

de Koster et al. [2] provide a nice overview of the order picking literature. Matson and White [13] provide a general overview of research in material handling, and van den Berg [18] provides a nice literature review of the planning and control of warehousing systems.

## 4.  STOCHASTIC ANALYSIS AND OPTIMAL DEPOT LOCATIONS

We now derive expressions for the expected walk distance required per order for each of our three configurations. In some expressions, it is convenient to define dummy locations at each end of the pick line where no picks occur with $q_0 = 1$ and $q_{n+1} = 1$. We let the random variable

$N =$ the total number of locations requiring picks
in an order.

Our method of modeling random orders will allow a null order, $N = 0$, to occur; however, we will condition all of our

expected walking distances on $N > 0$, non-null orders. We denote the probability of a non-null order by

$$\mathcal{P} = P(N > 0) = 1 - \prod_{i=1}^{n} q_i. \qquad (4.1)$$

## 4.1. Single-Depot

We now determine the expected walk distance per order when a single-depot is located along the pick line at location $k$ (see Fig. 3). The depot location $k$ can be any real value in $[1, n]$ (in fact, if the depot is actually located off the line, then dummy locations can be added from the depot to the line, each with pick probability set to zero). The walking required per order has two components: The distance (if any) required to move from the depot left to the leftmost pick and back to the depot, plus the distance (if any) required to move right to the rightmost pick and back to the depot. We define two random variables:

$L_k =$ the distance required to move left from $k$ to the leftmost pick,

and

$R_k =$ the distance required to move right from $k$ to the rightmost pick.

Now letting the random variable

$S_k =$ the total walk length per order for a single depot located at $k$,

we have

$$S_k = 2(L_k + R_k). \qquad (4.2)$$

We now define random variables for the location of the leftmost and rightmost pick of an order:

$L =$ the location of the leftmost pick,

and

$R =$ the location of the rightmost pick.

Therefore, we have

$$P(L = i) = p_i \prod_{j=1}^{i-1} q_j \quad \text{for any } i = 1, \ldots, n.$$

Similarly, we have

$$P(R = i) = p_i \prod_{j=i+1}^{n} q_j \quad \text{for any } i = 1, \ldots, n.$$

So we have

$$E[L_k] = \sum_{i=1}^{\lfloor k \rfloor} (k - i) P(L = i), \qquad (4.3)$$

and

$$E[R_k] = \sum_{i=\lceil k \rceil}^{n} (i - k) P(R = i). \qquad (4.4)$$

Now conditioning on non-null orders, $N > 0$, we have:

$$
\begin{aligned}
E[S_k | N > 0] &= \frac{E[S_k]}{\mathcal{P}} \\
&= \frac{2(E[L_k] + E[R_k])}{\mathcal{P}} \\
&= \frac{2}{\mathcal{P}} \left[ \sum_{i=1}^{\lfloor k \rfloor} (k - i) P(L = i) \right. \\
&\quad \left. + \sum_{i=\lceil k \rceil}^{n} (i - k) \, P(R = i) \right].
\end{aligned}
\qquad (4.5)
$$

## 4.2. Optimal Single-Depot Location

We now determine where to locate a single depot, $k^*$, that will minimize $E[S_k | N > 0]$, given by Eq. (4.5). We find that $k^*$ is a *median* location. In our case this means that $k^*$ is integral, corresponding exactly to a pick location. We choose $k^*$ so that the probability that the picker must walk to the left of the depot is equal to the probability that the picker must walk to the right.

THEOREM 4.1: The expected walk distance per order for a single depot is minimized when the depot is located at

$$
k^* = \min \left\{ k \left| \sum_{i=1}^{k} P(L = i | N > 0) \right. \right.
$$
$$
\left. - \sum_{i=k+1}^{n} P(R = i | N > 0) \geq 0 \right\}. \qquad (4.6)
$$

PROOF: This is a simple extension to the rectilinear facility location problem on a line (see [6]).

Here we seek to locate a new "facility" of minimal weighted distance to $n$ existing "facilities," where existing facility $i$ is located on the line at point $i$. The weight for existing facility $i$ is $P(L = i)$ if the new facility is located to the right of $i$ and $P(R = i)$ if located to the left. □

## 4.3. Dual-Depots

We now determine the expected walk distance per order with two depots, one located at $u$ and the other at $v$, where

$u \leq v$. With two depots we assume that the picker alternates between the depots (see Fig. 4).

We can consider the walking required per order to have three components: The distance required (if any) from the left depot to the leftmost pick and back to the left depot, the distance between the left depot and the right depot, and the distance required (if any) from the right depot to the right-most pick and back to the right depot. We let the random variable

$$D_{u,v} = \text{the total walk distance for dual depots at locations } u \text{ and } v.$$

So

$$D_{u,v} = 2L_u + (v - u) + 2R_v.$$

The expected walking distance per order for dual depots located at $u$ and $v$ along the pick line is given by

$$E[D_{u,v}|N > 0] = \frac{1}{\mathcal{P}}(2E[L_u] + (v-u) + 2E[R_v]). \quad (4.7)$$

We note that this expectation is a generalization of the single-depot case. That is

$$E[D_{k,k}|N > 0] = E[S_k|N > 0].$$

### 4.4. Optimal Dual-Depot Locations

We now determine the optimal depot locations $u^*$ and $v^*$ that will minimize $E[D_{u,v}]$, given by Eq. (4.7). As in the single-depot case, we again find that both $u^*$ and $v^*$ are median locations. Furthermore we can find the location $u^*$ that minimizes $E[L_u]$, and independently the location $v^*$ that minimizes $E[R_v]$.

THEOREM 4.2: The expected walk distance per order for dual depots is minimized when the left depot is located at

$$u^* = \min\left\{ u \,\middle|\, \sum_{i=1}^{u} P(L = i|N > 0) \geq \frac{1}{2} \right\}$$

and the right depot is located at

$$v^* = \max\left\{ v \,\middle|\, \sum_{i=v}^{n} P(R = i|N > 0) \geq \frac{1}{2} \right\}.$$

PROOF: The proof here follows in a similar way as the proof of Theorem 4.1. We find $u^*$ by considering an existing "facility" $i$ to have weight $2P(L = i)$ when the new facility is to the right of $i$, and we add a dummy existing facility at location $n$ with weight 1 representing the travel between depots. We then find $v^*$ in a similar fashion.    $\square$

We note that,

THEOREM 4.3: $u^* \leq k^* \leq v^*$.

PROOF: Since every order requires a pick at or to the left of $k$, a pick to the right of $k$, or both, then $\sum_{i=1}^{k} P(L = i|N > 0) + \sum_{i=k+1}^{n} P(R = i|N > 0) \geq 1$ for any $k$. By the definition of $k^*$ we know $\sum_{i=1}^{k^*} P(L = i|N > 0) \geq \sum_{i=k^*+1}^{n} P(R = i|N > 0)$, $\sum_{i=1}^{k^*} P(L = i|N > 0) \geq 1/2$, and $u^* \leq k^*$. An analogous argument shows $k^* \leq v^*$.    $\square$

OBSERVATION 1: As the picks become more dense, the optimal dual depot locations go to opposite pick ends ($u^* \rightarrow 1$ and $v^* \rightarrow n$), and as the picks become more sparse the optimal dual depot locations tend toward the optimal single depot location ($u^* \rightarrow k^*$ and $v^* \rightarrow k^*$).

### 4.5. No Depot

We now determine the expected walk distance per order with no depot. With no depot we assume that the picker alternates the direction he picks—picking one order left-to-right and the next right-to-left (see Fig. 5).

Thus the walking distance required per order has two basic components. First is the distance from the leftmost pick to the rightmost pick (the length of a pick run). Second, there is the distance from the end of one order to the start of the next: from the rightmost pick of one order to the rightmost pick of the next, or between leftmost picks of consecutive orders.

We let

$$T = \text{the total walk distance with no depot,}$$

and let

$$W = \text{the length of the pick run.}$$

We define the random variables for the distance from the end of one order to the beginning of the next:

$$B_R = \text{the distance between the rightmost pick of one order to the rightmost pick of another,}$$

and

$$B_L = \text{the distance between the leftmost pick of one order to the leftmost pick of another.}$$

So we have

$$T = W + \frac{1}{2}(B_L + B_R).$$

The expected length of a pick run is given by

$$E[W] = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} (|j - i| \, P(L = i) P(R = j)),$$

and conditioning out null orders gives

$$E[W|N > 0] = \frac{E[W]}{\mathcal{P}}.$$

For the expected distance between pick-ends we have

$$E[B_L] = \sum_{i=1}^{n} \sum_{j=1}^{n} |i - j| \, P(L = i) P(L = j),$$

and

$$E[B_R] = \sum_{i=1}^{n} \sum_{j=1}^{n} |i - j| \, P(R = i) P(R = j).$$

Letting $N_1$ and $N_2$ be the number of picks in two independent orders we have

$$P(N_1 > 0 \cap N_2 > 0) = \mathcal{P}^2,$$

and thus

$$E[B_L|N_1, N_2 > 0] = \frac{E[B_L]}{\mathcal{P}^2}$$

and,

$$E[B_R|N_1, N_2 > 0] = \frac{E[B_R]}{\mathcal{P}^2}.$$

The expected walk distance per order with no depot is given by

$$E[T] = E[W|N > 0] + \frac{1}{2}(E[B_L|N_1, N_2 > 0] \\ + E[B_R|N_1, N_2 > 0]). \qquad (4.8)$$

## 5. OPTIMAL ITEM LAYOUT

We examine how best to layout the product (items) when the depot location(s), if present, are fixed. We are given $n$ items that we wish to assign to $n$ locations. For each item $j$ we have a probability $c_j$ that no pick is required for item $j$ in a random order. We seek a complete assignment that sets each $q_i$ to a $c_j$, for $i, j = 1, \ldots, n$. For convenience we sequence items according to their $c_j$ values so that $c_1 \le c_2 \le \cdots \le c_n$.

### 5.1. Single-Depot Item Layout

In this section we assume that the depot location $k$ is integral. Jarvis and McDowell [9] solve for the best item layout for two special cases—when a single depot is at the start of the line, and when the depot is in the center. Our contribution here is a branch and bound procedure for the general case. In particular, we derive a result that characterizes the optimal layout that not only provides useful cuts in our branch and bound algorithm, but solves exactly and quickly the special cases of Jarvis and McDowell.

The conditional expected walking distance per order for a single depot at an integral location $k$, Eq. (4.5), can be rewritten as

$$E[S_k|N > 0]$$
$$= \frac{2}{\mathcal{P}} \left( (n - 1) - \sum_{i=1}^{k-1} \prod_{j=1}^{i} q_j - \sum_{i=1}^{n-k} \prod_{j=1}^{i} q_{n+1-j} \right). \quad (5.1)$$

If we consider the assignments to the left (or right) of the depot in an optimal layout, then a more popular item must be closer to the depot than a less popular item (this was also observed in [9]).

THEOREM 5.1: Any optimal layout with a single depot located at $k$ must assign items so that $q_1 \ge q_2 \ge \cdots \ge q_{k-1} \ge q_k$, and $q_k \le q_{k+1} \le \cdots \le q_{n-1} \le q_n$.

PROOF: To minimize Eq. (5.1) we wish to maximize the terms

$$\sum_{i=1}^{k-1} \prod_{j=1}^{i} q_j + \sum_{i=1}^{n-k} \prod_{j=1}^{i} q_{n+1-j}.$$

Consider the first term of items left of the depot. If one claims to have maximized this expression and yet there is some $q_i < q_j$ with $1 \le i < j \le k$, a simple swap will, in contradiction, increase it. An analogous argument holds for the second term, for items right of the depot. $\square$

Theorem 5.1 leads immediately to the following special case that was established in Jarvis and McDowell [9].

COROLLARY 5.2: When a single depot is located at the start of the line, $k = 1$, the layout where $q_1 = c_1 \le q_2 = c_2 \le \cdots \le q_n = c_n$ will minimize $E[S_1|N > 0]$.

Theorem 5.1 means that we can view our layout decision problem as a partitioning problem instead of a sequencing problem. Once we decide which $c_j$ to assign left of the depot

and which to the right, then the sequence is known. The number of possible partitions is

$$\binom{n-1}{k-1} = \frac{(n-1)!}{(k-1)!(n-k)!}. \qquad (5.2)$$

Our branch and bound scheme builds partial solutions by assigning the $c_j$ in sequence, from the largest to the smallest, to the left or to the right of the depot. Partial solutions are built up by first assigning locations at the ends of the line and then moving in toward the depot. So for example, a partial solution might be denoted by $(L, L, L, R, R, 0, 0, \ldots, 0)$, which indicates that the three largest $c_j$ ($c_n$, $c_{n-1}$, and $c_{n-2}$) are assigned left of the depot, the next two ($c_{n-3}$ and $c_{n-4}$) to the right of the depot, and the rest of the locations closest to the depot are unassigned or free.

For a particular partial solution, we let $f_L$ be the number of free locations to the left of the depot and $f_R$ be the number of free locations to the right. We let the probability that no pick is required so far in a partial solution to the left of the depot be given by

$$q_L = \prod_{i=0}^{k-1-f_L} q_i, \qquad (5.3)$$

and the probability that no pick is required to the right of the depot be given by

$$q_R = \prod_{i=0}^{n-k-f_R} q_{n+1-i}. \qquad (5.4)$$

The next theorem establishes that if one side has more free locations remaining than the other and a greater probability of no pick required, then the next largest $c_j$ should be assigned to that side.

THEOREM 5.3: If a partial solution has $f_L \geq f_R$ and $q_L \geq q_R$ then there exists an optimal way to layout the remaining free locations with the next largest $c_j$ assigned to the left side. And conversely, if $f_L \leq f_R$ and $q_L \leq q_R$ then there exists an optimal way to layout the remaining free locations with the next largest $c_j$ assigned to the right side.

PROOF: See Appendix, section A.2. □

Thus, Theorem 5.3 shows that $c_n$ (the least popular item) should be assigned to the side with a location farthest from the depot. Corollary 5.2 can be seen to follow from Theorem 5.3, and we will see that the other special case of a central depot given in [9] also follows from this theorem. More importantly, Theorem 5.3 generalizes the result of Jarvis and McDowell [9] and provides us a computationally useful cut in our

branch and bound scheme—allowing us to fathom any partial solution that violates the theorem.

The partitioning aspect of our problem and Theorem 5.1 leads to two natural heuristics. An alternating increasing layout (proposed by Jarvis and McDowell [9]) assigns $c_j$ in increasing value starting at the depot and alternating assignments to the left and right of the depot. When one side is full the other side is filled with the remaining $c_j$'s.

We also introduce an alternating decreasing layout that assigns $c_j$ in decreasing value starting at the end of the pick line and alternating assignments to the left and right of the depot. When a location next to the depot is first assigned then the remaining $c_j$ are filled into the remaining locations on the other side of the depot, moving towards the depot, until the depot is assigned $c_1$.

We formally define the two heuristics for $n$ locations and depot at $k$. Here we assume with no loss of generality that the integral depot location $k \leq n/2$:

**Alternating Increasing Layout:** Assign $c_j$ to $q_{k-(j-1)/2}$ for each $j = 1, 3, \ldots, 2k - 1$; to $q_{k+(j/2)}$ for each $j = 2, 4, \ldots, 2k$; and to $q_j$ for each $j = 2k, 2k + 1, \ldots, n$.

**Alternating Decreasing Layout:** Let $h = n - 2(k - 1)$. Assign $c_j$ to $q_{k+j-1}$ for each $j = 1, \ldots, h$; to $q_{k-(j+1-h)/2)}$ for each $j = h + 1, h + 3, \ldots, n - 1$; and to $q_{k-1+(j+h)/2)}$ for each $j = h + 2, h + 4, \ldots, n$.

We note that when the depot is centrally located, $k = \lfloor (n+1)/2 \rfloor$, then an AIL and ADL produce the same layout.

The following result established in Jarvis and McDowell also follows as a consequence of Theorem 5.3. Whenever $k$ is centrally located then an ADL or AIL is optimal.

THEOREM 5.4: When a single depot is located in the center of the line, at $\lfloor (n+1)/2 \rfloor$, then an ADL or AIL layout will minimize $E[S_k | N > 0]$.

PROOF: See Appendix, section A.3. □

When $k$ is located centrally or at the start of the line, our branch and bound scheme will return the optimal layout immediately by fathoming any partial solution that violates Theorem 5.3. For other $k$, the two heuristics, ADL and AIL, are used in our branch and bound scheme for upper bounds. Lower bounds for a partial solution are generated in a simple manner: If $c_j$ is the first (largest) unassigned $c_j$ we then assign $c_j, c_{j-1}, \ldots c_{j-f_L+1}$ values to the remaining free locations to the left of the depot, and $c_j, c_{j-1}, \ldots c_{j-f_R+1}$ values to the free locations to the right of the depot. The largest unassigned values are used on the remaining free locations on *both* sides of the depot.

**Table 1.** Run times (in cpu seconds on a 750 MHz PC) to obtain optimal layouts for a single depot at location $k$ with $n = 40$ locations for three different data sets, $r = 0.9, 0.7, 0.5$.

| | $n = 40$ Items | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $k$ | | | | | | | | | |
| | 1 | 2 | 4 | 6 | 8 | 10 | 12 | 14 | 16 | 18 | 20 |
| $r = .9$ | 0.01 | 0.01 | 0.03 | 0.52 | 5.22 | 37.71 | 150.19 | 395.35 | 131.30 | 18.87 | <0.01 |
| $r = .7$ | <0.01 | <0.01 | 0.09 | 4.59 | 93.51 | 827.52 | 2973.56 | 3363.54 | 680.27 | 7.09 | <0.01 |
| $r = .5$ | <0.01 | <0.01 | 0.10 | 4.08 | 52.26 | 235.48 | 393.81 | 189.23 | 24.93 | 0.46 | 0.01 |

Table 1 provides run times in CPU seconds for instances with $n = 40$ locations. Three different data sets were generated by setting each $c_j$ value to $1 - r^j$ for $r = 0.5, 0.7$ and $0.9$. Branching from a partial solution was halted whenever the lower bound was within 1% of the upper bound. The code was written in C and run on a 750 MHz PC. For each problem set, when the depot is near the end of the line (small $k$) the runs times are favorable because the number of possible partitions, by Eq. (5.2), is very small. When the depot is near the center of the line (large $k$) the number of possible partitions is the largest; however, the run times are small because Theorem 5.3 provides very effective fathoming. For intermediate depot locations the algorithm is the least effective as the number of partitions is fairly large and Theorem 5.3 is not as effective. When $n$ is increased beyond 40, the algorithm can still obtain solutions when $k$ is near the end or the center of the line; however, some instances with $n = 50$ fail to return an optimal solution after 10 h of CPU time.

### 5.2. Dual-Depot Item Layout

Here we assume that depot locations $u$ and $v$ are integral. Since a picker must always traverse between $u$ and $v$ for each order, then we place in any sequence the first $v - u + 1$ $c_j$ between the two depots. The remaining problem is then a single-depot layout problem with the remaining $c_j$ to be assigned to the left of $u$ and to the right of $v$. The AIL and ADL algorithms naturally adapt by only considering the remaining $c_j$ to be placed to the left of $u$ and right of $v$. Thus all our results for the single depot case apply for the dual-depot case.

We make clear that the solvable special case for a single, centrally located, depot (Theorem 5.4) corresponds to the dual-depot case when $u - 1 = n - v$, i.e. the two depots are equidistant from the ends of the pick line.

COROLLARY 5.5: When each depot is the same distance from its respective end of the pick line, then an ADL or AIL layout will minimize $E[D_{u,v}|N > 0]$.

### 5.3. No Depot Item Layout

We seek a layout to minimize Eq. (4.8),

$$E[T] = E[W|N > 0] + \frac{1}{2}(E[B_L|N_1, N_2 > 0] + E[B_R|N_1, N_2 > 0]).$$

To minimize $E[T]$ directly seems complex. However, what is desired is a compact pick run so that not only $E[W|N > 0]$ is small, but also $E[B_L|N_1, N_2 > 0]$ and $E[B_R|N_1, N_2 > 0]$. Thus an obvious heuristic is to assume a centrally located single depot and layout the items according to the ADL heuristic. This will minimize $E[W|N > 0]$, and thus is expected to perform well especially when the length of the pick run dominates the travel between the pick runs. We tested the ADL using enumeration on hundreds of randomly generated instances up to $n = 7$, and it always returned an optimal layout. It remains an open question if this heuristic is indeed an exact procedure to minimize $E[T]$.

## 6. SINGLE AND DUAL-DEPOT FULL DESIGN

We now consider the *full design* of how to simultaneously locate depots and layout the items. We know from Section 4.2 that it suffices to locate the depot at an integral location.

THEOREM 6.1: To minimize expected walk distance for a single depot, the depot can be placed in the center, at $\lfloor (n + 1)/2 \rfloor$, and the items laid out according to AIL (or equivalently ADL).

PROOF: To ease the exposition, we assume $n$ is odd. (If $n$ is even, then we can add a dummy location and a dummy item of zero pick probability. From Theorem 5.3, we know the dummy should always be located furthest from the depot, so this will not effect the expected walk distance of any viable design for the original instance.)

If one claims to have minimized the expected walk distance with the depot in the center, but items not in AIL, then from Theorem 5.4, an AIL will do no worse.

Suppose one claims an optimal design with depot location $\bar{k}$ not in the center with expected walk distance $E[S_{\bar{k}}]$. We can pad the left or right of the line with an even number, $d$, of dummy locations and items, each of zero pick probability, so that $\bar{k}$ is now in the center of an extended line. The expected walk distance of the extended line remains $E[S_{\bar{k}}]$, since one will never have to walk to the extended portions of the pick line for a dummy item. With the depot in the center of the extended line, we can by Theorem 5.4, lay out the items (including the dummy items) according to an AIL and the resulting expected walk distance will be no worse. By AIL, $d/2$ dummy items must be located at the left end of the pick line, and $d/2$ on the right end. We can now simply remove the dummy items with no change in the expected walk distance. Our depot is now centrally located with an AIL of the original items with expected walk distance no worse than the claimed optimal.                                               □

For dual depots the idea is the same—the dual depots should be centrally located and items laid out according to AIL. Except now one must decide how far apart the depots should be—and for this we can simply enumerate the $n - 1$ possibilities.

COROLLARY 6.2: To minimize the expected walk distance for dual depots we consider each of the $n - 1$ possible number of locations (including zero) between the two depots, and for each we locate the depots centrally and layout the items according to AIL. We then choose the best of the $n - 1$ cases.

## 7.  COMPARING THE CONFIGURATIONS

There is a clear relationship between the expected walking distance of each configuration.

THEOREM 7.1: For any set of $p_i$'s we have $E[S_1] \geq E[S_{k^*}] \geq E[D_{u^*,v^*}] \geq E[T]$.

PROOF: $E[S_1] \geq E[S_{k^*}]$ holds since $S_{k^*}$ is not constrained by the depot location. $E[S_{k^*}] \geq E[D_{u^*,v^*}]$ holds, since, $D_{u^*,v^*}$ is free to locate both depots at $k^*$. Also $E[D_{u^*,v^*}] \geq E[T]$ holds since the later need not visit an intermediate point between order pick ends.                            □

The configurations can differ significantly depending upon the $p_i$'s. Figure 6 shows the expected walk distance with $n = 10$ and $n = 50$ locations for each of our configurations (with the common configuration of a single depot at $k = 1$ included). For each data point we set each $p_i$ to the same $p$ value, and this value is then varied along the $x$-axis. The $E[S_{k^*}]$ and $E[D_{u^*,v^*}]$ curves are determined using the best

depot locations for each $p$ from Theorems 4.1 and 4.2. As the pick density, $p$, increases, the amount of walking increases for each configuration. (We note that we always condition our expected walking distances on $N > 0$, but omit the notation in this section.)

As the figures indicate clear limiting behavior exist. We state the following without proof.

LEMMA 7.2: For any fixed number of locations $n$, as the pick density increases we have the following limiting behavior:

$$\lim_{p \to 1} E[S_1] = 2(n - 1)$$

$$\lim_{p \to 1} E[S_{k^*}] = 2(n - 1)$$

$$\lim_{p \to 1} E[D_{u^*,v^*}] = n - 1$$

$$\lim_{p \to 1} E[T] = n - 1.$$

As the pick density increases, $p \to 1$, each order requires a visit to all locations. In this case a single depot anywhere along the line behaves poorly, forcing the picker to walk twice the length of the line for each order. On the other hand, a dual-depot configuration will tend to locate each depot at opposite ends of the line, so that the walking required per order will be one length of the line. And similarly, the no-depot configuration will tend to require the picker to walk one length of the line. We emphasize that two types of behavior emerge—any single-depot configuration requires twice the walk distance of either a dual or no-depot configuration.

We have the following limiting behavior as the pick density gets small.

LEMMA 7.3: For any fixed number of locations $n$, as the pick density decreases we have the following limiting behavior:

$$\lim_{p \to 0} E[S_1] = n - 1$$

$$\lim_{p \to 0} E[S_{k^*}] = \frac{n^2 - 1}{2n}$$

$$\lim_{p \to 0} E[D_{u^*,v^*}] = \frac{n^2 - 1}{2n}$$

$$\lim_{p \to 0} E[T] = \frac{n^2 - 1}{3n}.$$

As the pick density decreases (so that orders tend to have only a single pick), a single depot at the beginning performs
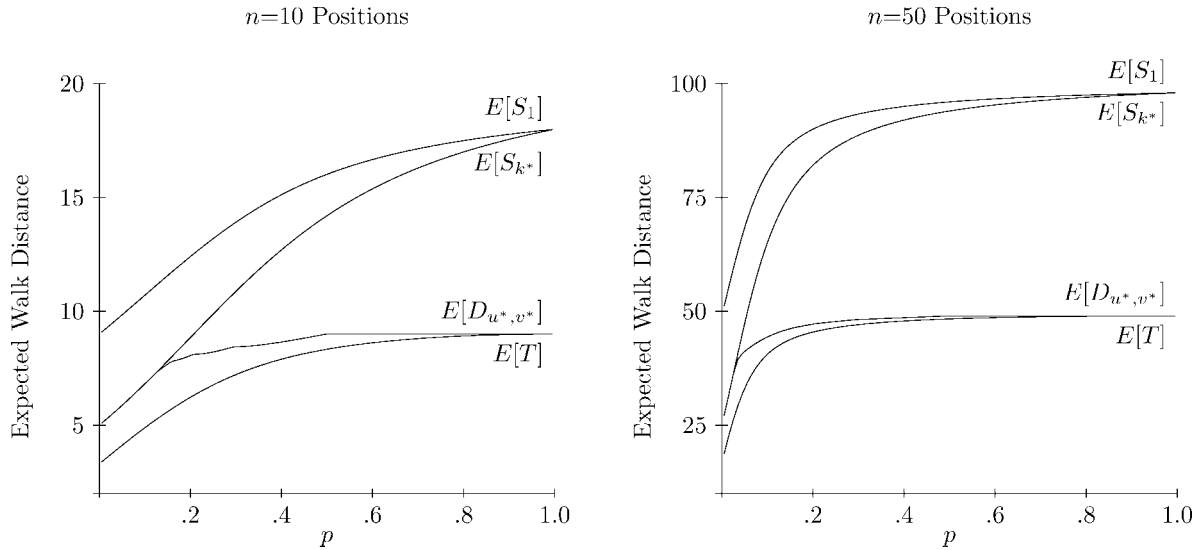
n=10 Positions                                              n=50 Positions



**Figure 6.** Here we show the expected walk distances for each configuration for both $n = 10$ and $n = 50$. Each item has the same the $p$ which we range along the $x$-axis.

very poorly, as pickers must, on average, walk to the center of the aisle and back. When a single depot is located in the center the picker must walk on average about one-fourth of the line and back. As the pick density gets very small, optimal dual depot locations will tend toward each other in the center of the line, and thus will behave the same as a single optimal depot. (This is because our dual-depot protocol forces the picker to alternate between the two depots. When the pick density is very small, dual depots would benefit by a more powerful and flexible algorithm.) Finally, the full technology

of a no-depot configuration gains a distinct advantage over all others, as it is able to better absorb the uncertainties of fetching small orders. So as the pick density goes to zero, three types of behavior emerge: a single depot at the beginning requires roughly twice the walking required by optimal single or optimal dual depots, and optimal single or optimal dual depots require 1.5 times the walking of a no-depot configuration.

A more realistic situation is one in which the $p_i$'s differ within a layout. Figure 7 compares the configurations
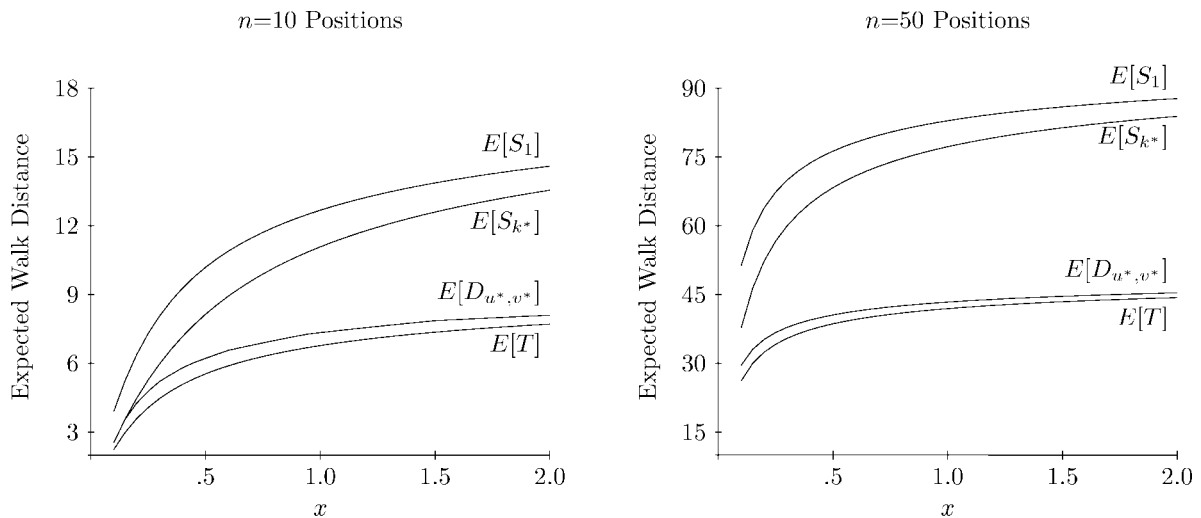
n=10 Positions                                              n=50 Positions



**Figure 7.** Here we show the expected walk distances for each configuration for both $n = 10$ and $n = 50$ when the $p_i$'s vary according to a parameter $x$, so that $\mathbf{p} = \left\{ 1, 1 - \left(\frac{1}{n}\right)^x, 1 - \left(\frac{2}{n}\right)^x, \ldots, 1 - \left(\frac{n-1}{n}\right)^x \right\}$.

for $n = 10$ and $n = 50$ with a parameter $x$ guiding how the $p$ values are dispersed. The set of $p_i$'s are given by $\mathbf{p} = \left\{ 1, 1 - \left(\frac{1}{n}\right)^x, 1 - \left(\frac{2}{n}\right)^x, \ldots, 1 - \left(\frac{n-1}{n}\right)^x \right\}$. So for example, when $x = 1$ the plots compare the four configurations for the problem instance when the $p_i$ values vary linearly from 1 down to $1/n$. When $x$ is close to 0, the $p_i$'s are skewed toward 0, and when $x$ is close to 2 the $p_i$'s are skewed closer to 1. For the single and dual depots we use the optimal full design, and for no-depot we use an ADL.

$E[S_1]$ and $E[S_{k*}]$ are very close, and thus for a typical distribution center in which the $p_i$'s vary, the location of a single depot has limited impact whenever one is free to layout the items optimally. More critical is the use of dual or no-depot technology, both of which outperform a single depot. The optimal use of dual depots is very powerful, nearly the same as a configuration with full technology investment. If space is an issue, so that the ability to add a conveyor in support of dual depots is prohibitive, then a no-depot configuration may be preferable.

In conclusion, the technology investment should be a function of the characteristics of the order stream. When pick densities are very large, orders tend to require visits to most pick locations, and thus it is important to avoid single-depot configurations. The technology of a conveyor or RF system to facilitate a dual-depot or no-depot configuration may be warranted. A dual-depot configuration performs very well overall, yielding significantly to the no depot case only when the pick densities are very small and uniform.

## 8. CONCLUSIONS

Order picking is commonly considered the most critical function in a supply chain; and within the warehouse the order picking function typically accounts for most of the operating costs. Furthermore, new e-commerce operations distinguish themselves from their traditional bricks-and-mortar competition in having to absorb the cost of the order picking function.

Discrete order picking is common due to its simplicity, reliability, and its ability to pick orders quickly upon receipt. However, it can also be wasteful, requiring considerable walking per pick. Technology is the first way to help reduce the walking required. The second is through the design—where to locate the depots (if any are used) and how best to layout the product.

Our work provides a manager a model to estimate the impact of different technology investments on his order picking system. We consider three levels of technology: (1) no technology (single-depot configuration); (2) conveyor technology (dual-depot configuration); (3) conveyor and RF technology (no-depot configuration). For each level

of technology we provide algorithms to optimally design the system—locate depot(s) and layout the product.

We find that the effectiveness of each level of technology depends on the characteristics of the order stream—many picks per order or very few. We introduce a new configuration (dual depots with conveyor technology) and find that it performs very well regardless of the type of order stream. The additional investment in RF technology is only warranted if the pick density is very small and uniform.

## APPENDIX: PROOF OF THEOREM 5.3

We will begin with a short lemma and some new notation.

### A.1. Some Technical Preliminaries

We will make use of the following simple lemma.

LEMMA A.1: If $x < y$ and $\alpha < \beta$ then $\alpha y + \beta x < \alpha x + \beta y$.

PROOF: $(\alpha x + \beta y) - (\alpha y + \beta x) = (y - x)(\beta - \alpha) > 0$.

We also will use the following notation when dealing with a sequence $s_1, s_2, \ldots$.

$$\mathbf{s}(i, j) = s_i + s_i s_{i+1} + s_i s_{i+1} s_{i+2} + \cdots + (s_i s_{i+1} \ldots s_j)$$
$$\text{for } i \leq j. \tag{A.1}$$

And we let the product of the terms of the sequence from $i$ to $j$, $i \leq j$, be denoted as

$$\dot{s}_{i,j} = s_i s_{i+1} \ldots s_j. \tag{A.2}$$

### A.2. Proof of Theorem 5.3

Consider a partial solution with $f_L$ free locations to the left of the depot and $f_R$ free locations to the right. We need only consider the one case in which $f_L \geq f_R$ and $q_L \geq q_R$. We relabel the free locations to the left of the depot, $q_{k-f_L}, q_{k-f_L+1}, \ldots, q_{k-1}$, as $a_1, a_2, \ldots a_{f_L}$ so that $a_1$ is furthest from the depot and $a_{f_L}$ is closest. We also relabel the free locations to the right of the depot as $b_{f_R}, b_{f_R-1}, \ldots, b_2, b_1$ where $b_{f_R}$ is closest to the depot and $b_1$ is farthest.

Rewriting Eq. (5.1), using Eqs. (5.3) and (5.4) and the notation of Eqs. (A.1) and (A.2), we seek to maximize

$$\mathbf{q}(1, k - 1 - f_L) + q_L \, \mathbf{a}(1, f_L) + \mathbf{q}(n, n - k - f_R)$$
$$+ q_R \, \mathbf{b}(1, f_R),$$

and after eliminating the assignments that are fixed we seek to maximize

$$q_L \mathbf{a}(1, f_L) + q_R \mathbf{b}(1, f_R). \tag{A.3}$$

Our theorem states that an optimal layout exists when the next largest $c_j$ is assigned to $a_1$ whenever $f_L \geq f_R$ and $q_L \geq q_R$. Suppose to the contrary that one claims to have completed the layout and maximized Eq. (A.3) with a larger $c_j$ assigned to $b_1$ instead of $a_1$. We will show that by swapping at

least $a_1$ and $b_1$ in such a proposed layout we can obtain a new layout that is no worse.

We consider two cases.

1. $\boldsymbol{a}(1, f_L) \geq \boldsymbol{b}(1, f_R)$ in the proposed layout.
   We rewrite Eq. (A.3) as

   $$q_L a_1 + q_L a_1 \boldsymbol{a}(2, f_L) + q_R b_1 + q_R b_1 \boldsymbol{b}(2, f_R). \qquad (A.4)$$

   Since $b_1 > a_1$, we know $\boldsymbol{a}(2, f_L) > \boldsymbol{b}(2, f_R)$, and thus, since $q_L \geq q_R$, we have from Lemma A.1, that Eq. (A.4) increases if we swap $a_1$ and $b_1$.

2. $\boldsymbol{a}(1, f_L) < \boldsymbol{b}(1, f_R)$ in the proposed layout.
   We let $j$ be the largest index so that $\dot{\boldsymbol{a}}_{1,j} < \dot{\boldsymbol{b}}_{1,j}$ (we know such an index exists since $a_1 < b_1$). If $j < f_R$, then we rewrite Eq. (A.3) as

   $$q_L \boldsymbol{a}(1, j) + q_L \dot{\boldsymbol{a}}_{1,j} \boldsymbol{a}(j+1, f_L) + q_R \boldsymbol{b}(1, j) + q_R \dot{\boldsymbol{b}}_{1,j} \boldsymbol{b}(j+1, f_R). \qquad (A.5)$$

   Since $f_R \leq f_L$, and by our choice of $j$, we have $\dot{\boldsymbol{b}}_{1,j} \boldsymbol{b}(j+1, f_R) < \dot{\boldsymbol{a}}_{1,j} \boldsymbol{a}(j+1, f_L)$. And thus, $\boldsymbol{b}(j+1, f_R) < \boldsymbol{a}(j+1, f_L)$.
   We rewrite our assumption $\boldsymbol{a}(1, f_L) < \boldsymbol{b}(1, f_R)$ as

   $$\boldsymbol{a}(1, j) + \dot{\boldsymbol{a}}_{1,j} \boldsymbol{a}(j+1, f_L) < \boldsymbol{b}(1, j) + \dot{\boldsymbol{b}}_{1,j} \boldsymbol{b}(j+1, f_R).$$

   But since $\dot{\boldsymbol{b}}_{1,j} \boldsymbol{b}(j+1, f_R) < \dot{\boldsymbol{a}}_{1,j} \boldsymbol{a}(j+1, f_L)$ we have $\boldsymbol{a}(1, j) < \boldsymbol{b}(1, j)$.

   So if we swap the $j$ values $b_1, \ldots, b_j$ and $a_1, \ldots, a_j$ in Equation (A.5) we get

   $$\begin{aligned} q_L \boldsymbol{b}(1, j) + q_L \dot{\boldsymbol{b}}_{1,j} \boldsymbol{a}(j+1, f_L) + q_R \boldsymbol{a}(1, j) \\ + q_R \dot{\boldsymbol{a}}_{1,j} \boldsymbol{b}(j+1, f_R), \end{aligned} \qquad (A.6)$$

   which, from Lemma A.1 is larger than Eq. (A.5).

If however, $j = f_R$ then we can show in a similar, but somewhat simpler, fashion that a swap of all $b_1, \ldots, b_{f_R}$ with $a_1, \ldots, a_{f_R}$ will result in a new layout that is no worse.

## A.3. Proof of Theorem 5.4

For ease of exposition we assume that $n$ is odd. (If $n$ is even we add a dummy location and dummy item of zero pick probability to the end of the line to make $n$ odd and the central depot at $k = \frac{n+1}{2}$. Neither the expected walk distance or an ADL/AIL layout of the original items is affected by the dummy item/location.)

Using the ADL heuristic, consider any iteration $i$ (the placement of $c_{n+1-i}$) whenever $i$ is odd. Then $f_L = f_R$. And $q_R = c_n c_{n-2} \cdots c_{n+3-i} \geq c_{n-1} c_{n-3} \cdots c_{n+2-i} = q_L$, since $c_j \geq c_{j-1}$ for each pairing with $j = n, n-2, \ldots, n+3-i$. So following ADL at any odd iteration and placing $c_{n+1-i}$ to the right of the depot maintains the requirements of Theorem 5.3.

Now consider iteration $i$ of the ADL heuristic whenever $i$ is even. Then $f_L > f_R$. And $q_R = c_n c_{n-2} c_{n-4} \cdots c_{n+2-i} \leq c_{n-1} c_{n-3} c_{n-5} \cdots c_{n+3-i} = q_L$, since $c_j \leq c_{j+1}$ for each pairing with $j = n-2, n-4, \ldots, n+2-i$, and $c_n c_{n-2} c_{n-4} \ldots c_{n+2-i} \leq c_{n-2} c_{n-4} \ldots c_{n+2-i} \leq c_{n-1} c_{n-3} c_{n-5} \ldots c_{n+3-i}$. So following ADL for any even iteration and placing $c_{n+1-i}$ to the left of the depot maintains the requirements of Theorem 5.3.

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